(4)

DNA-TR-86-247-V5

SOURCE REGION ELECTROMAGNETIC EFFECTS PHENOMENA
Volume V—Analytic Solutions for SREMP Environments

P. W. Van Alstine
L. Schlessinger
Pacific-Sierra Research Corporation
12340 Santa Monica Boulevard
Los Angeles, CA 90025-2587

25 April 1986

Technical Report

CONTRACT No. DNA 001-85-C-0235

Approved for public release; distribution is unlimited.

THIS WORK WAS SPONSORED BY THE DEFENSE NUCLEAR AGENCY UNDER RDT&E RMC CODES X326085466 V C 00003 25904D AND X326085466 V C 00005 25904D.

Prepared for
Director
DEFENSE NUCLEAR AGENCY
Washington, DC 20305-1000

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION: UNCLASSIFIED

1b. RESTRICTIVE MARKINGS:

2a. SECURITY CLASSIFICATION AUTHORITY: N/A since Unclassified

2b. DECLASSIFICATION / DOWNGRADING SCHEDULE: N/A since Unclassified

3. DISTRIBUTION / AVAILABILITY OF REPORT: Approved for public release; distribution is unlimited.

4. PERFORMING ORGANIZATION REPORT NUMBER(S): PSR Report 1588

5. MONITORING ORGANIZATION REPORT NUMBER(S): DNA-TR-86-247-V5

6a. NAME OF PERFORMING ORGANIZATION: Pacific-Sierra Research Corporation

6b. OFFICE SYMBOL (If applicable):

7a. NAME OF MONITORING ORGANIZATION: Director, Defense Nuclear Agency

6c. ADDRESS (City, State, and ZIP Code): 12340 Santa Monica Boulevard, Los Angeles, California 90025-2587

7b. ADDRESS (City, State, and ZIP Code): Washington, DC 20305-1000

8a. NAME OF FUNDING / SPONSORING ORGANIZATION:

8b. OFFICE SYMBOL (If applicable): RAAE/Tousley

9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: DNA 001-85-C-0235

8c. ADDRESS (City, State, and ZIP Code):

10. SOURCE OF FUNDING NUMBERS

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT ACCESSION NO.
62715H	V	C	DH251519

11. TITLE (Include Security Classification): SOURCE REGION ELECTROMAGNETIC EFFECTS PHENOMENA Volume V—Analytic Solutions for SREMP Environments

12. PERSONAL AUTHOR(S): Van Alstine, P. W.; Schlessinger, L.

13a. TYPE OF REPORT: Technical

13b. TIME COVERED: FROM 850501 TO 860330

14. DATE OF REPORT (Year, Month, Day): 860425

15. PAGE COUNT: 52

16. SUPPLEMENTARY NOTATION: This work was sponsored by the Defense Nuclear Agency under RDT&E RMC Codes X326085466 V C 00003 25904D and X326085466 V C 00005 25904D.

17. COSATI CODES

FIELD	GROUP	SUB-GROUP
20	3	
20	15	

18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number): Source Region Electromagnetic Pulse (SREMP); Electromagnetic Pulse (EMP)

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

We recently derived three-dimensional integral equations and integral solutions for the fields generated by a Compton current distribution with arbitrary time and space variations in the presence of conducting air and ground with arbitrary time variations in conductivity. Here we solve them analytically for a number of important SREMP cases which include 1) arbitrary three-dimensional currents in the presence of infinitely conductive ground, 2) equal air and ground conductivities, and 3) arbitrary one-dimensional (height-dependent) currents when the ground conductivity is a constant multiple of the (time-dependent) air conductivity. Using particular realistic forms for the Compton current, we explicitly evaluate our solutions to find the SREMP fields produced by them.

20. DISTRIBUTION / AVAILABILITY OF ABSTRACT: ☐ UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS

21. ABSTRACT SECURITY CLASSIFICATION: UNCLASSIFIED

22a. NAME OF RESPONSIBLE INDIVIDUAL: Sandra E. Young

22b. TELEPHONE (Include Area Code): (202) 325-7042

22c. OFFICE SYMBOL: DNA/CSTI

DD FORM 1473, 84 MAR — 83 APR edition may be used until exhausted. All other editions are obsolete.

SECURITY CLASSIFICATION OF THIS PAGE

UNCLASSIFIED

PREFACE

A goal of the Defense Nuclear Agency (DNA) electromagnetic pulse phenomenology program is the invention and development of new, improved methods to calculate source region electromagnetic pulse (SREMP) environments. As part of the ongoing contribution of Pacific-Sierra Research Corporation (PSR) to that program, this report supplies the mathematical development of new and improved techniques to calculate SREMP.

This report represents one area of the PSR research effort in SREMP. This document was prepared as one volume of the multivolume final technical report for DNA under contract DNA 001-85-C-0235. The technical monitor was MAJ William J. Farmer.

CONVERSION TABLE

Conversion factors for U.S. Customary to metric (SI) units of measurement

MULTIPLY → / TO GET ←	BY → / BY ←	TO GET → / DIVIDE ←
angstrom	1.000 000 X E -10	meters (m)
atmosphere (normal)	1.013 25 X E +2	kilo pascal (kPa)
bar	1.000 000 X E +2	kilo pascal (kPa)
barn	1.000 000 X E -28	meter^2 (m^2)
British thermal unit (thermochemical)	1.054 350 X E +3	joule (J)
calorie (thermochemical)	4.184 000	joule (J)
cal (thermochemical)/cm^2	4.184 000 X E -2	mega joule/m^2 (MJ/m^2)
curie	3.700 000 X E +1	*giga becquerel (GBq)
degree (angle)	1.745 329 X E -2	radian (rad)
degree Fahrenheit	$t_K = (t°f + 459.67)/1.8$	degree kelvin (K)
electron volt	1.602 19 X E -19	joule (J)
erg	1.000 000 X E -7	joule (J)
erg/second	1.000 000 X E -7	watt (W)
foot	3.048 000 X E -1	meter (m)
foot-pound-force	1.355 818	joule (J)
gallon (U.S. liquid)	3.785 412 X E -3	meter^3 (m^3)
inch	2.540 000 X E -2	meter (m)
jerk	1.000 000 X E +9	joule (J)
joule/kilogram (J/kg) (radiation dose absorbed)	1.000 000	Gray (Gy)
kilotons	4.183	terajoules
kip (1000 lbf)	4.448 222 X E +3	newton (N)
kip/inch^2 (ksi)	6.894 757 X E +3	kilo pascal (kPa)
ktap	1.000 000 X E +2	newton-second/m^2 (N-s/m^2)
micron	1.000 000 X E -6	meter (m)
mil	2.540 000 X E -5	meter (m)
mile (international)	1.609 344 X E +3	meter (m)
ounce	2.834 952 X E -2	kilogram (kg)
pound-force (lbs avoirdupois)	4.448 222	newton (N)
pound-force inch	1.129 848 X E -1	newton-meter (N·m)
pound-force/inch	1.751 268 X E +2	newton/meter (N/m)
pound-force/foot^2	4.788 026 X E -2	kilo pascal (kPa)
pound-force/inch^2 (psi)	6.894 757	kilo pascal (kPa)
pound-mass (lbm avoirdupois)	4.535 924 X E -1	kilogram (kg)
pound-mass-foot^2 (moment of inertia)	4.214 011 X E -2	kilogram-meter^2 (kg·m^2)
pound-mass/foot^3	1.601 846 X E +1	kilogram/meter^3 (kg/m^3)
rad (radiation dose absorbed)	1.000 000 X E -2	**Gray (Gy)
roentgen	2.579 760 X E -4	coulomb/kilogram (C/kg)
shake	1.000 000 X E -8	second (s)
slug	1.459 390 X E +1	kilogram (kg)
torr (mm Hg, 0° C)	1.333 22 X E -1	kilo pascal (kPa)

*The becquerel (Bq) is the SI unit of radioactivity; 1 Bq = 1 event/s.
**The Gray (Gy) is the SI unit of absorbed radiation.

TABLE OF CONTENTS

SECTION 1
INTRODUCTION

Recently, using Green's function techniques, we reduced a time-dependent SREMP problem to the solution of a set of integral equations for tangential values of electric and magnetic fields, E, B, on the ground surface [Van Alstine and Schlessinger, 1986].* When substituted into formal solutions for E and B, these surface values give a complete solution to the problem. The integral equations and formal solutions are an extension to three dimensions and arbitrary Compton currents, of earlier results using Laplace transform techniques for E and B fields depending on only one spatial variable (height) generated by Compton currents depending only on time. In this paper, we use the new three-dimensional equations to generate exact solutions for special cases that include SREMP fields that are produced in three dimensions above a perfectly conducting ground, SREMP fields that are produced in three dimensions in air and ground when air and ground conductivities are equal but time varying, and SREMP fields in one dimension when the time-dependent ground conductivity is a constant multiple of the time-dependent air conductivity. In addition, we show how the new three-dimensional equations contain our early one-dimensional results in two independent forms.

First we simplify our formal solutions for B and E by recasting them in a new form that makes clear their connection to underlying vector potentials. We then solve our integral equations for the case of an infinitely conductive ground and use the solutions to obtain B and E everywhere for that case.

Next, we solve our integral equations for the case of equal but time-dependent air and ground conductivities by using integral properties of the Green's function. We show that the exact solution

*Van Alstine, P., and L. Schlessinger, Source Region Electromagnetic Effects Phenomena, Vol. 4, New Methods for Determination of Three-Dimensional SREMP Environments, Pacific-Sierra Research Corporation, Report 1588, December 1986.

provided by our method is that given by the full-space vector-potential (which exists only in this case). Finally, we examine the (one-dimensional) case in which **B**, **E**, and the current **J** depend only on height (or depth) and time. We obtain the corresponding formal solutions for **B** and **E** as well as the integral equations that determine their values at the ground surface. We show that when **J** is further restricted to depend on time only, our integral equations and solutions reduce to those discovered through Laplace transform techniques in our early work [Schlessinger, 1984].* We also find that not only do the one-dimensional equations and solutions emerge as the remnants of our three-dimensional equations and solutions for currents and fields that only depend on height and time, but because of the structure of Maxwell's equations, the same one-dimensional equations and solutions govern the behavior of three-dimensional fields and currents averaged over transverse spatial variables (x and y).

We then solve our new one-dimensional equations for height-dependent **J** for the case in which the (time-dependent) ground conductivity is a constant multiple of the (time-dependent) air conductivity. We find an exact integral solution that is general enough to include previously found exact solutions for the cases of equal air and ground (time-dependent) conductivities, time-dependent air but infinite ground conductivities, and unequal but constant air and ground conductivities.

Using particular realistic forms for the Compton current, we evaluate our solutions for the fields in three dimensions above a perfect conductor and the fields in one dimension when time-dependent air and ground conductivities differ by a constant multiple. These explicit solutions provide insight into the general behavior of SREMP fields as well as analytic test solutions for comparison with numerical solutions for more general cases. In a subsequent report, we will present the numerical results for SREMP fields obtained from these solutions and compare them with those obtained by other methods.

---

*Schlessinger, L., Electromagnetic Effects Phenomena, Vol. 1, Analytical Solutions for SREMP Environments, Pacific-Sierra Research Corporation, Report 1437, November 1984 (subsequently published by the Defense Nuclear Agency, Washington, DC, as DNA TR-84-397-V1).

SECTION 2

SIMPLIFICATION OF FORMAL SOLUTION

In this section, we use vector identities to rewrite the integral expressions for **B** and **E** + **J**/σ of Van Alstine and Schlessinger [1986]* in two simpler forms. The first [Eqs. (7) and (9) below] makes clear the role played by image currents and is valid everywhere. The second [Eqs. (11) and (12)] displays **B** and **E** + **J**/σ as curls of vector quantities (everywhere but on the ground surface) and thus, explicitly demonstrates their divergencelessness (except at the ground). Both possess simple limits for infinitely conductive ground.

In Van Alstine and Schlessinger [1986],† we used Green's function techniques to derive a formal solution for **B** above and below a plane ground surface provided only that the (time-dependent) air and ground conductivities do not vary in space and that the displacement current is negligible in comparison with the conduction current in Maxwell's equations with Ohmic conduction current. That solution was given by:

$$\mathbf{B}\theta_B = -\nabla \times \int \frac{d^4x'}{\sigma'} G_0\mathbf{J} - (1 - 2\mathbf{kk}) \cdot \left(\overline{\nabla} \times \int \frac{d^4x'}{\sigma'} G_I\mathbf{J}\right) + 2\int_{-\infty}^{t} dt'\mathbf{\Omega} \cdot (\mathbf{n} \times \mathbf{E}) \;, \tag{1}$$

in which θ_B is $\theta(+z)$ and $\mathbf{n} = -\mathbf{k}$ for $z \geq 0$ (air), θ_B is $\theta(-z)$ and $\mathbf{n} = +\mathbf{k}$ for $z \leq 0$ (ground), $\mathbf{\Omega}$ is the integro-differential operator

*Van Alstine, P., and L. Schlessinger, Source Region Electromagnetic Effects Phenomena, Vol. 4, New Methods for Determination of Three-Dimensional SREMP Environments, Pacific-Sierra Research Corporation, Report 1588, December 1986.

†Ibid.

$$\Omega \cdot (n \times E) \equiv \int dS'[(n \times E)G_0]_{z'=0} - \nabla\nabla \cdot \int dS'[(n \times E)H]_{z'=0} , \qquad (2)$$

and G_0 and G_I are the infinite space scalar Green's function

$$G_0 = \left(- \frac{1}{2\sqrt{\pi L}}\right)^3 \exp\left[- \frac{(x - x')^2}{4L}\right] \theta(t - t') \qquad (3)$$

$\left(\text{where } L = \int_{t'}^{t} \frac{dt''}{\mu\sigma''}\right)$ and its image version parallel

$$G_I(x, y, z; x', y', z') = G_0(x, y, z; x', y', -z') , \qquad (4)$$

respectively. In Ω, H is the time-integrated Green's function:

$$H = \int_0^{L(t, t')} dL' G_0(L') = - \frac{1}{4\pi r} \operatorname{erfc}\left(\frac{r}{2L^{1/2}}\right) \theta(t - t') . \qquad (5)$$

where erfc is the complementary error function. A tilde over a vector indicates that the vector's z component is to be reversed in sign:

$$\widetilde{V} = (V_x, V_y, - V_z) .$$

The identity

$$(1 - 2kk) \cdot (\widetilde{A} \times B) = A_T \times kB_z - A_z k \times B_T - kk \cdot (A_T \times B_T) = - A \times \widetilde{B} , \qquad (6)$$

where the subscript T denotes tangential components, allows us to rewrite the second current term in Eq. (1) as

$$- (1 - 2kk) \cdot \left(\tilde{\nabla} \times \int \frac{d^4x'}{\sigma'} G_I J\right) = + \nabla \times \int \frac{d^4x'}{\sigma'} G_I \tilde{J} .$$

Then, Eq. (1) takes the simpler form:

$$B\theta_B = - \nabla \times \int \frac{d^4x'}{\sigma'} (G_0 J - G_I \tilde{J}) + 2 \int_{-\infty}^{t} dt' \, \Omega \cdot (n \times E) , \tag{7}$$

which makes clear the role of the second current term as the contribution to **B** of the "image current." A similar rearrangement of the formal solution for $E + J/\sigma$,

$$\left(E + \frac{J}{\sigma}\right) \theta_B = \frac{1}{\mu\sigma} \left[- \nabla \times \int \frac{d^4x'}{\sigma'} G_0 \nabla' \times J - (1 - 2kk) \cdot \tilde{\nabla} \times \int \frac{d^4x'}{\sigma'} G_I \nabla' \times J - 2 \int_{-\infty}^{t} dt' \, \Omega \cdot (n \times \dot{B}) \right] , \tag{8}$$

results in:

$$\left(E + \frac{J}{\sigma}\right) \theta_B = \frac{1}{\mu\sigma} \left[- \nabla \times \int \frac{d^4x'}{\sigma'} (G_0 \nabla' \times J - G_I \widetilde{\nabla' \times J}) - 2 \int_{-\infty}^{t} dt' \, \Omega \cdot (n \times \dot{B}) \right] . \tag{9}$$

Since the operator Ω is a perfect curl everywhere except on the ground surface

$$\Omega \cdot (n \times E) = - \nabla \times \nabla \times \int dS' \, n \times EH\Big|_{z'=0} - (n \times \dot{E}) \delta(z) \theta(t - t') , \tag{10}$$

except at $z = 0$ we can rewrite Eqs. (7) and (9) in terms of effective vector potentials:

$$B\theta_B = -\nabla \times \left[\int \frac{d^4x'}{\sigma'} (G_0 J - G_I \tilde{J}) + 2\nabla \times \int_{-\infty}^{t} dt' \int dS' \; n \times BH \Big|_{z'=0} \right] \tag{11}$$

$$\left(E + \frac{J}{\sigma}\right)\theta_B = -\nabla \times \frac{1}{\mu\sigma} \left[\int \frac{d^4x'}{\sigma'} (G_0 \nabla' \times J - G_I \widetilde{\nabla' \times J}) - 2\nabla \times \int_{-\infty}^{t} dt' \int dS' \; n \times \dot{B}H \Big|_{z'=0} \right] . \tag{12}$$

One may not use these forms to match solutions on the ground surface where second derivatives of H are singular. Eqs. (11) and (12) are inaccurate there; they should actually contain the δ-function subtraction included in Eq. (10) that makes their formally singular terms equal to (the nonsingular) Ω. Therefore, since a major goal of the present work is to solve the integral equations for B and E at the ground surface and to examine behavior of the fields B and E near the surface, we shall use the formal solutions Eqs. (7) and (9) that are accurate there in preference to Eqs. (11) and (12).

SECTION 3
INFINITE GROUND CONDUCTIVITY CASE

In the case of an infinitely conductive ground, the formal solution for B leads to an immediate solution of the Maxwell equations. If the ground conductivity is infinite, the electric field vanishes everywhere below the surface. Then continuity of the tangential electric field forces the tangential electric field to vanish at the surface as well:

$$k \times E\Big|_{z=0} = 0 \ . \qquad (13)$$

Then, the electric field-dependent term disappears from Eq. (7) giving B in the air directly in terms of the Compton current:

$$B_{>} = -\nabla \times \int_{>} \frac{d^4x'}{\sigma'} \left(G_0 J - G_I \tilde{J}\right) = -\nabla \times \int \frac{d^4x'}{\sigma'} \Gamma_E \cdot J \ , \qquad (14)$$

where $\Gamma_E = 1_T G_D + kkG_N$ is the "electric" Green's dyad, while G_D and G_N are the scalar Dirichlet and Neumann Green's functions, respectively. Ampere's law ($E = -J/\sigma + \nabla \times B/\mu\sigma$) then leads directly to a solution for E in the air:

$$E_{>} = -\frac{J}{\sigma_{>}} + \frac{1}{\mu\sigma_{>}}\left[\nabla^2 \int_{>} \frac{d^4x'}{\sigma'} \left(G_0 J - G_I \tilde{J}\right) - \nabla\nabla \cdot \int_{>} \frac{d^4x'}{\sigma'} \left(G_0 J - G_I \tilde{J}\right)\right] . \qquad (15)$$

*As in Van Alstine, P., and Schlessinger, Source Region Electromagnetic Effects Phenomena, Vol. 4, New Methods for Determination of Three-Dimensional SREMP Environments, Pacific-Sierra Research Corporation, Report 1588, December 1986, the subscripts > and < denote versions of physical quantities evaluated above and below ground, respectively.

But, in the air,

$$\frac{\nabla^2}{\mu\sigma_>} G_0 = \partial_t G_0 + \delta(x - x')\delta(t - t') \tag{16}$$

and

$$\frac{\nabla^2}{\mu\sigma_>} G_I = \partial_t G_I \quad \text{for } z,\ z' \geq 0 \ , \tag{17}$$

so that after integration by parts in the time [and use of $\partial_t G = -(\sigma'/\sigma)\partial_{t'} G$],

$$E_> = \frac{1}{\sigma_>} \int_> d^4x' (G_D \dot{J}_T + kG_N \dot{J}_z) + \frac{1}{\mu\sigma_>} \nabla \int_> \frac{d^4x'}{\sigma'} (\nabla' G_0 \cdot J - \tilde{\nabla}' G_I \cdot \tilde{J}) \ . \tag{18}$$

By using Γ_E and integrating the inner gradients in the final term of (Eq. 18) by parts, we obtain the compact form:

$$E_> = \frac{1}{\sigma_>} \int_> d^4x' \ \Gamma_E \cdot \dot{J} - \frac{\nabla}{\mu\sigma_>} \int_> \frac{d^4x'}{\sigma'} G_D \nabla' \cdot J \ . \tag{19}$$

Equations (14) and (19) give the complete solution for the fields B and E everywhere above an infinitely conductive ground in terms of an arbitrary Compton current distributed in the air.

SECTION 4

EQUAL AIR AND GROUND CONDUCTIVITY CASE

When the (time-dependent) air and ground conductivities are equal, Green's theorem applied to all of space immediately gives the solution for **B**:

$$\mathbf{B} = -\nabla \times \int_{>+<} \frac{d^4x'}{\sigma'} G_0 \mathbf{J} \ , \tag{20}$$

and hence (through Ampere's law) **E**:

$$\mathbf{E} = \frac{1}{\sigma} \int_{>+<} d^4x' \ G_0 \dot{\mathbf{J}} - \frac{\nabla}{\mu\sigma} \int_{>+<} \frac{d^4x'}{\sigma'} G_0 \ \nabla' \cdot \mathbf{J} \tag{21}$$

(when **J** is continuous at the ground surface). Note that in this case, the Helmholtz theorem implies that the divergenceless vector field **B** is given both in air and ground as the curl of a single vector potential which may be taken to be:

$$\mathbf{A} = -\int_{>+<} \frac{d^4x'}{\sigma'} G_0 \mathbf{J} \ . \tag{22}$$

Thus, since we already know its solution, this case serves as a check on the validity of our integral solutions and integral equations. It provides us with an opportunity to use integral identities (that may be important in the solution of the general problem) to effectively solve the integral equation that is the heart of our method. Once we have solved the integral equation, in the rest of this section we shall demonstrate that our method produces the result contained in Eqs. (20) and (21).

When we set the transverse components of **B** in air and ground as given by Eq. (7) equal to each other at the ground surface in order to satisfy the appropriate boundary condition (assuming the absence of surface currents), we obtain the integral equation:

$$\int_{-\infty}^{t} dt' \Omega \cdot (k \times E)_{>+<,\ z=0,T} = - \left[\nabla \times \int_{>-<} \frac{d^4x'}{\sigma'} G_0 J \right]_{z=0,T} , \qquad (23)$$

in the general case. But when $\sigma_> = \sigma_< = \sigma(t)$, the air and ground Ω terms become equal so that:

$$2 \int_{-\infty}^{t} dt' \Omega \cdot (k \times E)_{z=0,T} = - \left[\nabla \times \int_{>-<} \frac{d^4x'}{\sigma'} G_0 J \right]_{z=0,T} . \qquad (24)$$

That is, when we explicitly write out the meanings of the surface values of terms on the right-hand side as inherited from matching transverse B in air and ground:

$$2 \int_{-\infty}^{t} dt' \Omega \cdot (k \times E)_{z=0,T}$$

$$= - \left(\lim_{z \to 0+} \nabla \times \int_{>} \frac{d^4x'}{\sigma'} G_0 J - \lim_{z \to 0-} \nabla \times \int_{<} \frac{d^4x'}{\sigma'} G_0 J \right)_T . \qquad (25)$$

Now, we are supposed to use the boundary information contained in Eq. (25) to determine the electric field term in Eq. (7), the formal solution for B. There are, in fact, at least two and perhaps three ways to do this. The one that we shall use here (because it is tailor-made for the present case) uses integral identities implied by Green's theorem to turn Eq. (25) directly into an expression for the electric field term in Eq. (7). A second method uses a special order of integral identities, differentiations, and limits to invert the scalar Green's functions contained in Ω. In one dimension this operation determines the surface value of the tangential electric field, which when substituted into Eq. (7) directly gives the solution for **B**. However, in three dimensions, it leads to a more complicated vector object which is itself sufficient to determine the electric field term of Eq. (7). Or, we may try even in three dimensions to solve for the surface value of tangential **E** eventually using it in Eq. (7) to give **B**.

Here, when we apply the integral identities as detailed in Appendix B to Eq. (25), we find that they move the Ω term off the z = 0 surface to variable z, giving

$$2\int_{-\infty}^{t} dt'\,\Omega \cdot (\mathbf{k} \times \mathbf{E})_T = + \left(\nabla \times \int_{>} \frac{d^4x'}{\sigma'}\, G_I \tilde{\mathbf{J}}_{>} + \nabla \times \int_{<} \frac{d^4x'}{\sigma'}\, G_0 \mathbf{J}_{<} \right)_T \quad , \tag{26}$$

which is just what we need to determine **B**. Substitution of this into the transverse part of the solution for **B** as given by Eq. (7) implies that

$$\mathbf{B}_{>T} = \left[- \nabla \times \int_{>} \frac{d^4x'}{\sigma'} \left(G_0 \mathbf{J}_{>} - G_I \tilde{\mathbf{J}}_{>} \right) \right]_T - \left[\nabla \times \int_{>} \frac{d^4x'}{\sigma'}\, G_I \tilde{\mathbf{J}}_{>} + \nabla \times \int_{<} \frac{d^4x'}{\sigma'}\, G_0 \mathbf{J}_{<} \right]_T \quad . \tag{27}$$

so that

$$\mathbf{B}_{>T} = \left[- \nabla \times \int_{>+<} \frac{d^4x'}{\sigma'}\, G_0 \mathbf{J} \right]_T \quad . \tag{28}$$

Having found $\mathbf{B}_T$ over all space, we could now evaluate it on the ground surface and use the result in the z component of the integral solution Eq. (7) to find B_z over all space. However, the fact that **B** is divergenceless everywhere provides us with a method to use Eq. (28) for $\mathbf{B}_T$ to determine B_z directly without intermediate surface limits. That is,

$$\nabla \cdot \mathbf{B}_{>} = 0 \tag{29}$$

implies that

$$\partial_z B_{>z} = - \nabla_T \cdot \mathbf{B}_{>T} , \tag{30}$$

so that

$$B_{>z} = \int_z^{\infty} d\xi \; \nabla_T \cdot \mathbf{B}_{>T} . \tag{31}$$

But Eq. (28) states that $\mathbf{B}_T$ is the transverse part of the curl of a vector field:

$$\mathbf{B}_{>T} = (\nabla \times \mathbf{A})_T , \tag{32}$$

where

$$\mathbf{A} = - \int_{>+<} \frac{d^4x'}{\sigma'} G_0 \mathbf{J} . \tag{33}$$

When we substitute Eq. (32) into Eq. (31)

$$B_{>z} = \int_z^{\infty} d\xi \; \nabla_T \cdot (\nabla \times \mathbf{A})_T = \int_z^{\infty} d\xi \left[\nabla_T \cdot (\nabla_T \times A_z \mathbf{k}) + \nabla_T \cdot (\mathbf{k}\partial_z \times \mathbf{A}_T) \right] , \tag{34}$$

we find that the integrand is a perfect differential in z, so that

$$B_{>z} = - \int_z^{\infty} \mathbf{k}\partial_z \cdot [\nabla_T \times \mathbf{A}_T] = \mathbf{k} \cdot (\nabla_T \times \mathbf{A}_T) . \tag{35}$$

Thus, using Eq. (33) for $\mathbf{A}$, this becomes

$$\mathbf{B}_{>z} = -\mathbf{k}\cdot\left(\nabla\times\int_{>+<}\frac{d^4x'}{\sigma'}G_0\mathbf{J}\right). \qquad (36)$$

Therefore, everywhere in the air $\mathbf{B}$ is given by

$$\mathbf{B}_{>} = -\nabla\times\int_{>+<}\frac{d^4x'}{\sigma'}G_0\mathbf{J}\ . \qquad (37)$$

Repetition of this procedure for $\mathbf{B}$ in the ground leads to the right-hand side of Eq. (37) as well so that:

$$\mathbf{B} = -\nabla\times\int_{>+<}\frac{d^4x'}{\sigma'}G_0\mathbf{J}\ , \qquad (38)$$

everywhere above and below the ground surface, in agreement with the solution Eq. (20) obtained from Green's theorem for all space as the curl of the vector potential given by Eq. (22) or Eq. (33).

Since Eq. (38) gives $\mathbf{B}$ everywhere, Ampere's law ($\mathbf{E} = -\mathbf{J}/\sigma + \nabla\times\mathbf{B}/\mu\sigma$) determines $\mathbf{E}$ everywhere:

$$\mathbf{E} = -\frac{\mathbf{J}}{\sigma} - \frac{1}{\mu\sigma}\left(-\nabla^2\int_{>+<}\frac{d^4x'}{\sigma'}G_0\mathbf{J} + \nabla\nabla\cdot\int_{>+<}\frac{d^4x'}{\sigma'}G_0\mathbf{J}\right), \qquad (39)$$

which simplifies to:

$$\mathbf{E} = \frac{1}{\sigma}\int_{>+<}d^4x'\ G_0\dot{\mathbf{J}} - \frac{\nabla}{\mu\sigma}\int_{>+<}\frac{d^4x'}{\sigma'}G_0\ \nabla'\cdot\mathbf{J} \qquad (40)$$

(when $\mathbf{J}$ is continuous at the ground surface), after use of the Green's function equation and integrations by parts in space and time.

SECTION 5

REDUCTION TO ONE-DIMENSIONAL SOLUTION

In earlier work [Schlessinger, 1984],* Laplace transform techniques were used to develop a version of our solution method applicable to "one-dimensional" problems in which the (three-dimensional) **B** and **E** fields depend only on the single spatial variable z. Such solutions are physically important for two reasons. First, for (time-dependent) conductivities and Compton currents of interest there may be periods of time for which the the fields vary in the horizontal spatial coordinates only over distances much larger than the effective diffusion length (or skin depth):

$$D(t, t') = 2(L)^{1/2} . \qquad (41)$$

For such periods, the one-dimensional solutions will provide good approximations to the true fields. Second, as we shall show, in our case the Maxwell equations for these one-dimensional **B** and **E** fields are identical in form to the partial differential equations obeyed by **B**, **E**, and **J** averaged over the transverse spatial variables. Thus, even when Compton currents and the fields they produce are strongly varying in the transverse variables, those fields have an average (over the transverse variables) behavior that is predicted by the one-dimensional problem.

When the Compton current **J** and the **B** and **E** fields produced by it depend only on z and t, each of the spatial integrals in the formal solution for **B**, Eq. (7), becomes an integral of the transverse spatial dependence of the Green's function over all x and y, i.e.,

*Schlessinger, L., Electromagnetic Effects Phenomena, Vol. 1, Analytical Solutions for SREMP Environments, Pacific-Sierra Research Corporation, Report 1437, November 1984 (subsequently published by the Defense Nuclear Agency, Washington, DC, as DNA TR-84-397-V1).

$$
\begin{aligned}
B\theta_B = -\nabla \times \int \frac{dt'}{\sigma'}\, dz' \Bigg[\left(\int dS' G_0\right) J(z', t') - \left(\int dS' G_I\right) \tilde{J}(z', t')\Bigg] \\
+ 2\int_{-\infty}^{t} dt' \Bigg\{\Bigg[\left(\int dS' G_0\right) n \times E(z', t')\Bigg]_{z'=0} \\
- \nabla\nabla \cdot \Bigg[\left(\int dS' H\right) n \times E(z', t')\Bigg]_{z'=0}\Bigg\} . \qquad (42)
\end{aligned}
$$

Since the Green's function factorizes,

$$
\begin{aligned}
G_0 &= \left(-\frac{1}{2\sqrt{\pi L}}\right)^3 e^{-\frac{(x - x')^2}{4L}}\, \theta(t - t') \\
&= \left(-\frac{1}{2\sqrt{\pi L}}\right) e^{-\frac{(z - z')^2}{4L}} \left(-\frac{1}{2\sqrt{\pi L}}\right)^2 e^{-\frac{(x_T - x_T')^2}{4L}}\, \theta(t - t') ,
\end{aligned}
\qquad (43)
$$

and since

$$
\int_{-\infty}^{+\infty} d\xi' + \left(\frac{1}{2\sqrt{\pi L}}\right) e^{-\frac{(\xi - \xi')^2}{4L}} = 1 , \qquad (44)
$$

the transverse spatial integral of G is just

$$
\int dS' G_0 = G_{01} , \qquad (45)
$$

where G_1 is the one-dimensional diffusion Green's function that depends only on z and z'. The consequences of Eq. (45) are that

$$
\int dS' G_0 f(z', t') = G_{01} f(z', t') , \qquad (46)
$$

$$\partial_z \int dS' G_0 f(z', t') = \partial_z G_{01} f(z', t') \ , \tag{47}$$

and

$$\nabla\nabla_T \cdot \int dS' [(\mathbf{n} \times \mathbf{E})H]_{z'=0} = 0 \ . \tag{48}$$

Thus, in Eq. (7)

$$\boldsymbol{\Omega} \cdot (\mathbf{n} \times \mathbf{E}) = \int dS' [\mathbf{n} \times \mathbf{E} G_0]_{z'=0} = [\mathbf{n} \times \mathbf{E} G_{01}]_{z'=0} \tag{49}$$

so that Eq. (7) becomes

$$\mathbf{B}\theta_B = - k\partial_z \times \iint \frac{dt'}{\sigma'} dz' (G_{01}\mathbf{J} - G_{I1}\tilde{\mathbf{J}}) + 2\int_{-\infty}^{t} dt' [\mathbf{n} \times \mathbf{E} G_{01}]_{z'=0} \ . \tag{50}$$

Since the right-hand side of Eq. (50) is transverse, an immediate consequence is that B_z vanishes everywhere. Since J_z does not contribute to the current term of Eq. (50), **B** can be rewritten as

$$\mathbf{B}\theta_B = - k\partial_z \times \iint \frac{dt'}{\sigma'} dz' G_{D1}\mathbf{J}_T + 2\int_{-\infty}^{t} dt' [\mathbf{n} \times \mathbf{E} G_{01}]_{z'=0} \ . \tag{51}$$

Similarly, performing the transverse spatial integrals in the integral equation [Eq. (23)] that results from matching transverse magnetic fields in air and ground at $z = 0$ yields the integral equation:

$$\int_{-\infty}^{t} dt' \ \mathbf{k} \times \mathbf{E}\Big|_{z'=0} G_{01}(z=0, z'=0)_{>+<} = - k\partial_z \times \int_{>-<} \frac{dt'}{\sigma'} dz' G_{01}(z=0)\mathbf{J}_T \ . \tag{52}$$

Likewise, if we perform the transverse spatial integrals in the formal solution for $\mathbf{E} + \mathbf{J}/\sigma$ in Eq. (9), we find

$$\left(\mathbf{E} + \frac{\mathbf{J}}{\sigma}\right) \theta_B = \frac{1}{\mu\sigma} \left[- \mathbf{k}\partial_z \times \iint \frac{dt'}{\sigma'} dz' G_{01} \mathbf{k}\partial_{z'} \times \mathbf{J} - 2 \int_{-\infty}^{t} dt' [\mathbf{n} \times \dot{\mathbf{B}} G_{01}]_{z'=0} \right], \tag{53}$$

which immediately implies that $E_z = -J_z / \sigma$ everywhere. When we carry out the same integrations in the integral equation that results from matching transverse components of $\mathbf{E}$ as given by Eq. (9) at the ground surface, we find

$$\left[\frac{1}{\mu\sigma} \int_{-\infty}^{t} dt' \mathbf{k} \times \dot{\mathbf{B}}\Big|_{z'=0} G_{01}(z=0,\ z'=0)\right]_{>+<}$$

$$= \left[\frac{\mathbf{J}_T}{2\sigma} + \frac{1}{\mu\sigma} \mathbf{k}\partial_z \times \iint \frac{dt'}{\sigma'} dz' G_{01} \mathbf{k}\partial_{z'} \times \mathbf{J}\right]_{>-<,z=0}. \tag{54}$$

The formal solutions for $\mathbf{B}$ and $\mathbf{E}$ and the integral equations contained in Eqs. (51) through (54) are just those that we would have obtained had we applied one-dimensional Green's function techniques directly to the one-dimensional problem. They emerge here whenever the transverse spatial variations of $\mathbf{J}$ and hence $\mathbf{B}$ and $\mathbf{E}$ are small enough over distances on the order of the diffusion scale that the transverse dependence of the Green's function can be integrated away.

However, there is a more general significance to these equations. If we return to the formal solutions for $\mathbf{B}$ and $\mathbf{E}$ in the general case as given by Eqs. (7) and (9), and average them over the transverse spatial variables using Eq. (45) and its consequences, we find:

$$\langle \mathbf{B}\rangle\theta_B = -k\partial_z \times \iint \frac{dt'}{\sigma'}\, dz' G_{D1}\langle J_T\rangle + 2\int_{-\infty}^{t} dt'[\mathbf{n} \times \langle \mathbf{E}\rangle G_{01}]_{z'=0} , \tag{55}$$

and

$$\left(\langle \mathbf{E}\rangle + \frac{\langle \mathbf{J}\rangle}{\sigma}\right)\theta_B = \frac{1}{\mu\sigma}\left[-k\partial_z \times \iint \frac{dt'}{\sigma'}\, dz' G_{D1} k\partial_{z'} \times \langle J\rangle - 2\int_{-\infty}^{t} dt'[\mathbf{n} \times \langle \dot{\mathbf{B}}\rangle G_{01}]_{z'=0}\right], \tag{56}$$

where < > indicates an average over transverse spatial variables. These equations are identical in form to Eqs. (51) and (53), with transverse spatial averages of **B**, **E**, and **J** replacing their one-dimensional versions. Thus, the one-dimensional solutions have a more general physical significance as transverse spatial averages of three-dimensional solutions. In fact, this is a direct consequence of the structure of Maxwell's equations for our SREMP problem. When the displacement current can be neglected, Maxwell's equations read:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon} \tag{57}$$

$$\nabla \times \mathbf{E} + \dot{\mathbf{B}} = 0 \tag{58}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{59}$$

$$\nabla \times \mathbf{B} = \mu(\mathbf{J} + \sigma\mathbf{E}) . \tag{60}$$

Their transverse spatial averages then become:

$$\partial_z\langle E_z\rangle = \frac{\langle \rho\rangle}{\varepsilon} \tag{61}$$

$$k\partial_z \times \langle \mathbf{E}_T\rangle + \langle \dot{\mathbf{B}}\rangle = 0 \tag{62}$$

$$\partial_z\langle B_z\rangle = 0 \tag{63}$$

$$\mathbf{k}\partial_z \times \langle \mathbf{B}_T\rangle = \mu(\langle \mathbf{J}\rangle + \sigma\langle \mathbf{E}\rangle) \ , \tag{64}$$

when $\sigma = \sigma(t)$ is not a function of spatial variables. Note that Eqs. (61) through (64) are identical in form to the Maxwell equations [Eqs. (57) through (60)] themselves with averaged **B**, **E**, **J** replacing one dimensional versions of **B**, **E**, **J** that depend only on z and t.

Equations (51) through (54) are more general than those originally obtained through the use of Laplace transform techniques [Schlessinger, 1984].* That work assumed that the Compton current J depends only on the time. In that situation one can perform an integration by parts in the current term of Eq. (51) so that one can use the fact that $\partial/\partial z J(t) = 0$. Then Eq. (51) becomes:

$$\mathbf{B}\theta_B = 2\int dt' \left[G_{01}\mathbf{n} \times \left(\mathbf{E} + \frac{\mathbf{J}}{\sigma'}\right)\right]_{z'=0} \ , \tag{65}$$

which can be rewritten using Ampere's law as:

$$\mathbf{B}\theta_B = \pm\, 2\int \frac{dt'}{\mu\sigma'} \left(\partial_{z'}\mathbf{B}_T G_{01}\right)_{z'=0\pm} \ . \tag{66}$$

Similarly, Eq. (53) becomes:

$$\left(\mathbf{E} + \frac{\mathbf{J}}{\sigma}\right)\theta_B = -\frac{2}{\mu\sigma}\int_{-\infty}^{t} dt'\left(\mathbf{n} \times \dot{\mathbf{B}}G_{01}\right)_{z'=0} \ . \tag{67}$$

Equation (66) for **B** is equivalent to Eq. (16) of Schlessinger [1984]* (after some rearrangement of the earlier work). When evaluated on the

---

*Schlessinger, L., Electromagnetic Effects Phenomena, Vol. 1, Analytical Solutions for SREMP Environments, Pacific-Sierra Research Corporation, Report 1437, November 1984 (subsequently published by the Defense Nuclear Agency, Washington, DC, as DNA TR-84-397-V1).

ground surface $z = 0$, Eqs. (66) and (67) reduce directly to the integral relations Eqs. (17) through (20) of Schlessinger [1984].* If one carries out the same integration by parts and uses $\partial/\partial z J(t) = 0$ in the integral equations [Eqs. (52) and (54)], one finds:

$$\left\{ \int \frac{dt'}{\mu\sigma'} \left[\partial_{z'} B_T \Big|_{z'=0} G_{01}(z=0,\ z'=0) \right] \right\}_{>+<} = 0 \ , \tag{68}$$

and

$$\left\{ \frac{2}{\sigma} \int dt' \left[k \times \dot{B} \Big|_{z'=0} G_{01}(z=0,\ z'=0) \right] \right\}_{>+<} = \mu \left(\frac{J_{T>}}{\sigma_>} - \frac{J_{T<}}{\sigma_<} \right)_{z=0} \ , \tag{69}$$

respectively. Equation (69) is identical to the integral equation [Eq. (25)] of Schlessinger [1984],† while continuity of transverse E in Ampere's law:

$$E_T \Big|_{z=0+} = \frac{k\partial_z \times B_>}{\mu\sigma_>} \Big|_{z=0} - \frac{J_{T>}}{\sigma_>} \Big|_{z=0} = E_T \Big|_{z=0-} = \frac{k\partial_z \times B_<}{\mu\sigma_<} \Big|_{z=0} - \frac{J_{T<}}{\sigma_<} \Big|_{z=0} \ , \tag{70}$$

converts Eq. (68) into the integral equation [Eq. (24)] of Schlessinger [1984].‡ Consequently, all of the analytic and semianalytic results of that previous work may be obtained from the more general one-dimensional solutions and integral equations of the present work [Eqs. (51) through (54)].

---

*Schlessinger, L., Electromagnetic Effects Phenomena, Vol. 1, Analytical Solutions for SREMP Environments, Pacific-Sierra Research Corporation, Report 1437, November 1984 (subsequently published by the Defense Nuclear Agency, Washington, DC, as DNA TR-84-397-V1).

†Ibid.

‡Ibid.

SECTION 6
GENERALIZED ONE-DIMENSIONAL EXACT SOLUTION

In order to obtain a solution to the Maxwell equations [Eqs. (57 through (60)] in air and ground in the general one-dimensional case, one must first solve one of the integral equations [e.g., Eq. (52)] for a surface field and substitute the result into an integral solution [e.g., Eq. (51)] to obtain a field throughout space. In general, this requires the numerical solution of the integral equation. However, for a special class of cases that include the one-dimensional infinite ground conductivity case, equal air and ground conductivity case, and the constant but unequal air and ground conductivity case, one can use the integral equation [Eq. (52)] to obtain an analytic solution for B (and hence for E). In the general case, the integral equation for the surface value of E is:

$$\int_{-\infty}^{t} dt' k \times E\Big|_{z'=0} G_{01}(z=0,\ z'=0)_{>+<} = -k\partial_z \times \iint_{>-<} \frac{dt'}{\sigma'} dz' G_{01}(z=0) J_T \ . \tag{52}$$

In order to use this information to determine B, we must transform it into the term

$$2\int_{-\infty}^{t} dt' [n \times EG_{01}]_{z'=0} \ , \tag{71}$$

in the formal one-dimensional solution for B, Eq. (51). We can do this either by solving Eq. (52) for $k \times E|_{z=0}$ by inverting $\int dt' G_{01}(z=0,\ z'=0)_{>+<}$, or by using the integral identity Eq. (126) of Appendix B (as we did in Sec. 4) to turn the left side of Eq. (52) into Eq. (71). We choose the second method here. Note that the integral identity Eq. (126) of Appendix B tells us how to perform integrations involving products of two Green's functions that solve the same differential equation. But, if we multiply Eq. (52) by either $\partial_z G_{01>}$ or $\partial_z G_{01<}$, we will be forced to deal with integrals involving both $G_>$ and $G_<$. Our integral identities will apply to these only if

$$G_{01<}(z=0,\ z'=0) = \alpha G_{01>}(z=0,\ z'=0)\ , \tag{72}$$

where α is a constant in space and time. This occurs when

$$L_< = L_>/\alpha^2\ , \tag{73}$$

which implies that

$$\alpha^2 = \sigma_<(t)/\sigma_>(t) = \text{const.}\ , \tag{74}$$

for all t. In this case, Eq. (52) becomes:

$$\begin{aligned}(1 + \alpha)\int_{-\infty}^{t} dt'k \times E\Big|_{z'=0} G_{01>}(z=0,\ z'=0) \\ = -\lim_{z\to 0+} k\partial_z \times \iint_{>} \frac{dt'}{\sigma'_>}\, dz' G_{01>} J_{T>} \\ + \lim_{z\to 0-} k\partial_z \times \iint_{<} \frac{dt'}{\sigma'_<}\, dz' G_{01<} J_{T<}\ ,\end{aligned} \tag{75}$$

whose left side is entirely given in terms of $G_>$.

Before we can use our integral identities, however, since they refer to Green's functions whose common z argument has been made to vanish from the same side of the plane z = 0, we must rewrite the term on the right of Eq. (75) that depends on $J_<$ in terms of a $z \to 0$ limit from above (just as we did in Appendix B):

$$\lim_{z\to 0-} k\partial_z \times \iint_< \frac{dt'}{\sigma'_<} dz' G_{01<} J_{T<} = -\lim_{z\to 0+} k\partial_z \times \iint_< \frac{dt'}{\sigma'_<} dz' G_{I1<} J_{T<}(z')$$

$$= -\lim_{z\to 0+} k\partial_z \times \iint_> \frac{dt'}{\sigma'_<} dz' G_{01<} J_{T<}(-z') \; . \tag{76}$$

Because of the form of Eq. (76), we are faced with a further problem. Even though we have assumed in Eq. (72) that $G_<$ is proportional to $G_>$ when both z and z' have been set to zero, this is no longer true when as in Eq. (76) z' is variable. In fact,

$$G_{01<} = \left(-\frac{1}{2\sqrt{\pi L_<}}\right) e^{-\frac{(z-z')^2}{4L_<}} \theta(t-t')$$

$$= \alpha\left(-\frac{1}{2\sqrt{\pi L_>}}\right) e^{-\frac{(z-z')^2\alpha^2}{4L_>}} \theta(t-t') \; . \tag{77}$$

However, if we use Eq. (77) in Eq. (76), we find that because the z variable is eventually set to 0, $G_<$ may still be turned into $G_>$ by scaling z' so that $z'_{new} = \alpha\, z'_{old}$. That is,

$$\lim_{z\to 0+} \partial_z \iint_> \frac{dt'}{\sigma'_<} dz' G_{01<} J_{T<}(-z')$$

$$= -\alpha \lim_{z\to 0+} \iint_> \frac{dt'}{\sigma'_>} \frac{dz'}{\alpha^2} \partial_{z'} \left[\left(-\frac{1}{2\sqrt{\pi L_>}}\right) e^{-\frac{(z-z')^2\alpha^2}{4L_>}} \theta(t-t')\right] J_{T<}(-z')$$

$$= -\frac{1}{\alpha}\lim_{z\to 0+}\iint_{>}\frac{dt'}{\sigma'_{>}}\,dz'\partial_{z'}\left[\left(-\frac{1}{2\sqrt{\pi L_{>}}}\right)e^{-\frac{(z-z')^2}{4L_{>}}}\theta(t-t')\right]J_{T<}\left(\frac{-z'}{\alpha}\right)$$

$$= \frac{1}{\alpha}\lim_{z\to 0+}\partial_z\iint_{>}\frac{dt'}{\sigma'_{>}}\,dz'G_{01>}J_{T<}\left(\frac{-z'}{\alpha}\right)\ . \tag{78}$$

The integral equation [Eq. (75)] has now become:

$$(1+\alpha)\int_{-\infty}^{t} dt'k \times E\Big|_{z'=0} G_{01>}(z=0,\ z'=0)$$

$$= \lim_{z\to 0+}\left\{-k\partial_z \times \iint_{>}\frac{dt'}{\sigma'_{>}}\,dz'G_{01>}J_{T>} - \frac{k\partial_z}{\alpha}\times\iint_{>}\frac{dt'}{\sigma'_{>}}\,dz'G_{01>}J_{T<}\left(\frac{-z'}{\alpha}\right)\right\}\ . \tag{79}$$

We multiply both sides by $-2\partial_{z'}G_{>}|_{z'=0}$, integrate over $\int dt'/\mu\sigma'_{>}$, and use the integral identities Eqs. (129) and (128) of Appendix B. The net result is that

$$(1+\alpha)\int_{-\infty}^{t} dt'[k \times EG_{01>}]_{z'=0}$$

$$= k\partial_z \times \iint_{>}\frac{dt'}{\sigma'_{>}}\,dz'G_{I1>}J_{T>} + \frac{1}{\alpha}\,k\partial_z \times \iint_{>}\frac{dt'}{\sigma'_{>}}\,dz'G_{I1>}J_{T<}\left(\frac{-z'}{\alpha}\right)\ . \tag{80}$$

Setting $z'\to -z'$ in the second term on the right side and dividing both sides by $(1+\alpha)$ then converts this into:

$$\int_{-\infty}^{t} dt'[\mathbf{k} \times \mathbf{E}G_{01>}]_{z'=0} = \frac{1}{1+\alpha} \, \mathbf{k}\partial_z \times \iint_{>} \frac{dt'}{\sigma'_{>}} \, dz' G_{I1>} \mathbf{J}_{T>}$$

$$+ \frac{1}{\alpha(1+\alpha)} \, \mathbf{k}\partial_z \times \iint_{<} \frac{dt'}{\sigma'_{>}} \, dz' G_{01>} \mathbf{J}_{T<} \left(\frac{z'}{\alpha} \right) , \qquad (81)$$

an expression for the surface **E**-field term in the formal one-dimensional solution for **B**, Eq. (51). If we substitute Eq. (81) into Eq. (51), we find:

$$\mathbf{B}_{>} = - \mathbf{k}\partial_z \times \iint_{>} \frac{dt'}{\sigma'_{>}} \, dz' G_{01>} \mathbf{J}_{T>} + \mathbf{k}\partial_z \times \iint_{>} \frac{dt'}{\sigma'_{>}} \, dz' G_{I1>} \mathbf{J}_{T>}$$

$$- \frac{2}{(1+\alpha)} \, \mathbf{k}\partial_z \times \iint_{>} \frac{dt'}{\sigma'_{>}} \, dz' G_{I1>} \mathbf{J}_{T>}$$

$$- \frac{2}{\alpha(1+\alpha)} \, \mathbf{k}\partial_z \times \iint_{<} \frac{dt'}{\sigma'_{>}} \, dz' G_{01>} \mathbf{J}_{T<} \left(\frac{z'}{\alpha} \right) \qquad (82)$$

in which the two image terms may be combined to give:

$$\mathbf{B}_{>} = - \mathbf{k}\partial_z \times \iint_{>} \frac{dt'}{\sigma'_{>}} \, dz' \left[G_{01>} + \frac{(1-\alpha)}{(1+\alpha)} \, G_{I1>} \right] \mathbf{J}_{T>}$$

$$- \frac{2}{\alpha(1+\alpha)} \, \mathbf{k}\partial_z \times \iint_{<} \frac{dt'}{\sigma'_{>}} \, dz' G_{01>} \mathbf{J}_{T<} \left(\frac{z'}{\alpha} \right) , \qquad (83)$$

for **B** in the air. The analogous procedure applied to **B** in the ground yields:

$$B_< = -k\partial_z \times \iint_< \frac{dt'}{\sigma'_<} dz' \left[G_{01<} + \frac{\left(1 - \frac{1}{\alpha}\right)}{\left(1 + \frac{1}{\alpha}\right)} G_{I1<} \right] J_{T<}$$

$$- \frac{2}{\frac{1}{\alpha}\left(1 + \frac{1}{\alpha}\right)} k\partial_z \times \iint_> \frac{dt'}{\sigma'_<} dz' G_{01<} J_{T>}(\alpha z') \ . \qquad (84)$$

By using Ampere's law ($E = -J/\sigma + \nabla \times B/\mu\sigma$), we compute the corresponding E fields in air and ground:

$$E_> = -k \frac{J_{z>}}{\sigma_>} + \frac{1}{\sigma_>} \left\{ \iint_> dt'dz' \left[G_{01>} + \frac{(1 - \alpha)}{(1 + \alpha)} G_{I1>} \right] \dot{J}_{T>} \right.$$

$$\left. + \frac{2}{\alpha(1 + \alpha)} \iint_< dt'dz' \ G_{01>} \dot{J}_{T<} \left(\frac{z'}{\alpha} \right) \right\} , \qquad (85)$$

and

$$E_< = -k \frac{J_{z<}}{\sigma_<} + \frac{1}{\sigma_<} \left\{ \iint_< dt'dz' \left[G_{01<} + \frac{\left(1 - \frac{1}{\alpha}\right)}{\left(1 + \frac{1}{\alpha}\right)} G_{I1<} \right] \dot{J}_{T<} \right.$$

$$\left. + \frac{2}{\frac{1}{\alpha}\left(1 + \frac{1}{\alpha}\right)} \iint_> dt'dz' \ G_{01<} \dot{J}_{T>} \ (\alpha z') \right\} , \qquad (86)$$

respectively. The one-dimensional cases in which the ground conductivity is infinite, the (time-dependent) air and ground conductivities are equal, and the air and ground conductivities are unequal but constant all satisfy $\sigma_< / \sigma_> =$ const. and hence produce B and E fields given by Eqs. (83) through (86). These solutions generalize to time dependent σ's (whose ratio is constant) and J(z,t) the solutions found in Schlessinger [1984]* for slowly-varying conductivities and z-independent J's.

---

*Schlessinger, L., Electromagnetic Effects Phenomena, Vol. 1, Analytical Solutions for SREMP Environments, Pacific-Sierra Research Corporation, Report 1437, November 1984 (subsequently published by the Defense Nuclear Agency, Washington, DC, as DNA TR-84-397-V1).

SECTION 7

EXPLICIT SOLUTIONS FOR SPECIALIZED COMPTON CURRENT DISTRIBUTIONS

In order to investigate important features of the SREMP fields yielded by our equations, we evaluate them for two Compton current distributions of physical interest, one in the one-dimensional situation of Sec. 6 and one in the space above an infinitely conductive ground as treated in Sec. 3. First, for the one-dimensional case in which air and ground conductivities differ by a constant multiple, we choose a current pulse that exists only at one time, that is constant in z in the air, and that is exponentially attenuated below the ground surface:

$$J(z,t) = \begin{cases} \hat{J}_{>}\delta(t) & z > 0 \\ \hat{J}_{<}\delta(t)e^{z/\lambda} & z < 0 \; . \end{cases} \tag{87}$$

When we use this **J** in the appropriate solution for **B** in air, Eq. (83), we can immediately perform the time integrals to obtain:

$$\begin{aligned} B_{>} = &- \frac{1}{\sigma_{>}} k\partial_z \times \int_{>} dz' \left[G_{01>}(t) + \left(\frac{1-\alpha}{1+\alpha}\right) G_{I1>}(t) \right] \hat{J}_{>} \\ &- \frac{2}{\alpha(1+\alpha)\sigma_{>}} k\partial_z \times \int_{<} dz' G_{01>}(t) \hat{J}_{<} e^{\frac{z'}{\alpha\lambda}} \end{aligned} \tag{88}$$

in which L(t,t') has been evaluated by the δ-function at t' = 0 so that

$$G_0(t) \equiv G_0(L(t,\, t'))|_{t'=0} = - \frac{1}{2\sqrt{\pi L(t)}} e^{-\frac{(z - z')^2}{4L(t)}} \; . \tag{89}$$

Since the air current was assumed to be constant in z, the two terms in Eq. (88) that depend on the air current have integrands that are perfect differentials and can be integrated immediately to give:

$$B_{>} = -\frac{k}{\sigma_{>}} \times \hat{J}_{>} G_{01>}(z'=0,\ t) + \frac{(1 - \alpha)}{(1 + \alpha)}\, k \times \frac{\hat{J}_{>}}{\sigma_{>}}\, G_{I1>}(z'=0,\ t)$$

$$- \frac{2}{\alpha(1 + \alpha)\sigma_{>}}\, k \times \hat{J}_{<} \partial_z \int_{<} dz' G_{01>}(t) e^{\frac{z'}{\alpha\lambda}} \ . \qquad (90)$$

But $G_I(z'=0) = G_0(z'=0)$ so that the integrated terms can be combined to yield:

$$B_{>} = -\frac{2\alpha}{1 + \alpha}\, k \times \frac{\hat{J}_{>}}{\sigma_{>}}\, G_{01>}(z'=0,\ t)$$

$$- \frac{2}{\alpha(1 + \alpha)\sigma_{>}}\, k \times \hat{J}_{<}\ \partial_z \int_{<} dz' G_{01>}(t) e^{\frac{z'}{\alpha\lambda}} \ . \qquad (91)$$

Now, the integral inside the z derivative in the ground current term in Eq. (91) can be rewritten as:

$$I = \int_{<} dz' \left(-\frac{1}{2\sqrt{\pi L_{>}}}\right) e^{-\frac{(z - z')^2}{4L_{>}}}\, e^{\frac{z'}{\alpha\lambda}}$$

$$= \int_{<} dz' \left(-\frac{1}{2\sqrt{\pi L_{>}}}\right) e^{-\frac{z^2}{4L_{>}}}\, e^{-\frac{1}{4L_{>}}\left[z'^2 - 2\left(z + \frac{2L_{>}}{\lambda\alpha}\right) z'\right]} \ . \qquad (92)$$

When we complete the square in the argument of the z'-dependent exponential it becomes:

$$I = \left(-\frac{1}{2\sqrt{\pi L_>}}\right) e^{-\frac{z^2}{4L_>}} e^{+\frac{1}{4L_>}\left[z + \frac{2L_>}{\lambda\alpha}\right]^2} \int_{-\infty}^{0} dz'\, e^{-\frac{1}{4L_>}\left[z' - \left(z + \frac{2L_>}{\lambda\alpha}\right)\right]^2} \tag{93}$$

so that a change of variable $\xi = \dfrac{z' - \left(z + \frac{2L_>}{\lambda\alpha}\right)}{2L_>^{1/2}}$ produces:

$$I = -\frac{1}{2}\, e^{\left[\frac{z}{\lambda\alpha} + \frac{L_>}{(\lambda\alpha)^2}\right]} \operatorname{erfc}\left(\frac{z}{2L_>^{1/2}} + \frac{L_>^{1/2}}{\lambda\alpha}\right). \tag{94}$$

When we put Eq. (94) back into Eq. (91), carry out the z derivative in the final term of Eq. (91), and combine terms, we find:

$$B_> = \frac{\theta(t)}{(1 + \alpha)}\left[\frac{\alpha}{\sqrt{\pi L_>}}\, \frac{e^{-\frac{z^2}{4L_>}}}{\sigma_>}\left(\mathbf{k} \times \hat{J}_> - \frac{\mathbf{k} \times \hat{J}_<}{\alpha^2}\right)\right.$$

$$\left. + \frac{1}{\lambda\alpha^2}\, \frac{\mathbf{k} \times \hat{J}_<}{\sigma_>}\, e^{\frac{z}{\lambda\alpha}}\, e^{\frac{L_>}{(\lambda\alpha)^2}} \operatorname{erfc}\left(\frac{z}{2L_>^{1/2}} + \frac{L_>^{1/2}}{\lambda\alpha}\right)\right]. \tag{95}$$

Note that as a result of the scaled structure of the solution given by Eq. (83), throughout the ground current term in Eq. (95) the current attenuation length λ appears in the combination $\Lambda = \alpha\lambda$. Thus,

$$B_> = \theta(t)\, \frac{\alpha}{(1 + \alpha)}\left\{\frac{e^{-\frac{z^2}{4L_>}}}{\sigma_>\sqrt{\pi L_>}}\, \mathbf{k} \times \left[\hat{J}_> - \frac{\hat{J}_<}{\alpha^2}\right] + \mathbf{k} \times \frac{\hat{J}_<}{\sigma_<}\, \frac{f(z)}{\Lambda}\right\}, \tag{96}$$

where

$$f(z) \equiv e^{\frac{z}{\Lambda}} e^{\frac{L_>}{\Lambda^2}} \operatorname{erfc}\left(\frac{z}{2L_>^{1/2}} + \frac{L_>^{1/2}}{\Lambda}\right).$$

Similar steps for this special current lead to

$$B_< = \theta(t) \frac{\alpha}{(1+\alpha)} \left\{ \frac{\alpha\, e^{-\frac{z^2}{4L_<}}}{\sigma_< \sqrt{\pi L_<}}\, k \times \left[\hat{J}_> - \frac{\hat{J}_<}{\alpha^2}\right] + k \times \frac{\hat{J}_<}{\sigma_<} \frac{1}{2\lambda} \left[\left(1 + \frac{1}{\alpha}\right) f(\alpha z) - \left(1 - \frac{1}{\alpha}\right) f(-\alpha z)\right]\right\}, \tag{97}$$

for B in the ground;

$$E_> = -\frac{\hat{J}_>}{\sigma_>} \delta(t) - \frac{\theta(t)}{\mu\sigma_>} \frac{\alpha}{(1+\alpha)} \left\{ \frac{1}{\sigma_> \sqrt{\pi L_>}} \left(\hat{J}_> - \frac{\hat{J}_<}{\alpha^2}\right) T^{\partial_z} e^{-\frac{z^2}{4L_>}} - \frac{1}{\Lambda} \frac{\hat{J}_{<T}}{\sigma_<} \left(\frac{e^{-\frac{z^2}{4L_>}}}{\sqrt{\pi L_>}} - \frac{f(z)}{\Lambda}\right)\right\}, \tag{98}$$

for E in the air; and

$$\mathbf{E}_< = -\frac{\hat{\mathbf{J}}_<}{\sigma_<} e^{\frac{z}{\lambda}} \delta(t) - \frac{\theta(t)}{\mu\sigma_<} \frac{\alpha}{(1 + \alpha)} \left\{ \frac{\alpha}{\sigma_< \sqrt{\pi L_<}} \left[\hat{\mathbf{J}}_> - \frac{\hat{\mathbf{J}}_<}{\alpha^2} \right] T \partial_z e^{-\frac{z^2}{4L_<}} \right.$$

$$\left. - \frac{1}{\lambda} \frac{\hat{\mathbf{J}}_< T}{\sigma_<} \left(\frac{e^{-\frac{z^2}{4L_<}}}{\sqrt{\pi L_<}} - \frac{1}{2\lambda} \left[\left(1 + \frac{1}{\alpha}\right) f(\alpha z) + \left(1 - \frac{1}{\alpha}\right) f(-\alpha z) \right] \right) \right\}, \quad (99)$$

for $\mathbf{E}$ in the ground.

We can also obtain useful solutions for the field in the space above a perfect conductor when we choose a simple current form. For this case, we choose a Compton current density which is only in the radial direction and is a shell expanding at the speed of light--that is

$$\mathbf{J}(\mathbf{x}, t) = J(r, t)\hat{\mathbf{r}} = J_0 R_0^{\,3} \frac{\delta(r - ct)}{r^2} \exp(-r/\lambda)\, \hat{\mathbf{r}} . \quad (100)$$

We also assume that the air conductivity is given by the simple form:

$$\sigma(t) = \sigma_0 \exp(-t/t_0) . \quad (101)$$

With this form we find that

$$L(t, t') = \frac{t_0}{\mu\sigma(t)} [1 - \exp -(t - t')/t_0] . \quad (102)$$

To evaluate the magnetic field in this case we use Eq. (14). Integrating by parts and using the fact that **curl** $(\mathbf{J}) = 0$ we find

$$\mathbf{B} = -2 \int \frac{dt'}{\sigma'} \int dS' G_0 (\mathbf{k} \times \mathbf{J}) \Big|_{z'=0} . \quad (103)$$

Using the radial nature of the current in Eq. (103) we can perform the angular integrals to obtain

$$B = \frac{1}{2\sqrt{\pi}}\,\hat{\phi}\int_{-\infty}^{t}\frac{dt'}{\sigma' L^{3/2}}\int_{0}^{\infty}\rho d\rho \exp - \left(\frac{r^2 + \rho^2}{4L}\right) I_1\left(\frac{r\rho\sin\theta}{2L}\right) J(\rho, t') , \tag{104}$$

where I_1 is the modified Bessel function of the first order. Using the form for J given in Eq. (100) we obtain

$$B = \frac{\mu J_0 R_0^{\ 3}}{2\sqrt{\pi} c t_0 d}\,\hat{\phi}\int_0^1 \frac{dx \exp\,(t/t_0(x - 1) - ctx/\lambda) \exp - \left(\frac{r^2 + c^2t^2x^2}{4d^2\ell}\right)}{\ell^{3/2}x} I_1\left(\frac{rctx\sin\theta}{2d^2\ell}\right) \tag{105}$$

where

$$d^2 = \frac{t_0}{\mu\sigma(t)} ,$$

$$\ell = 1 - \exp\left(- \frac{t(1 - x)}{t_0}\right) = \frac{L}{d^2} .$$

We can easily obtain the solution for constant air conductivity from Eq. (105) by setting $t_0 \to \infty$ so that $\sigma(t) = \sigma_0 = \text{const}$. We find:

$$B_c = \frac{\mu J_0 R_0^{\ 3}}{2\sqrt{\pi}\ ct}\left(\frac{\mu\sigma_0}{t}\right)^{1/2}\hat{\phi}\int_0^1 dx\,\frac{\exp\left(-\frac{ctx}{\lambda}\right)}{x(1 - x)^{3/2}}\ \exp\left[-\frac{\mu\sigma_0}{4t}\,\frac{r^2 + c^2t^2x^2}{(1 - x)}\right] I_1 \tag{106}$$

where the argument of I_1 is

$$\left(\frac{\mu\sigma_0 rcx\sin\theta}{2(1 - x)}\right).$$

Ampere's Law $\left(\mathbf{E} = -\frac{\mathbf{J}}{\sigma} + \frac{1}{\mu\sigma}\nabla \times \mathbf{B}\right)$ then immediately gives us the $\mathbf{E}$ field corresponding to the $\mathbf{B}$ field of Eq. (104) generated by a radial current above a perfect conductor:

$$\mathbf{E} = -\frac{\mathbf{J}}{\sigma} + \frac{1}{4\mu\sigma\sqrt{\pi}}\int_{-\infty}^{t} \frac{dt'}{\sigma' L^{5/2}}\int_{0}^{\infty} \rho d\rho J(\rho, t') \exp -\left(\frac{r^2 + \rho^2}{4L}\right)$$

$$\cdot \left[\hat{\mathbf{r}}\rho\cos\theta I_0 - \hat{\boldsymbol{\theta}}(\rho\sin\theta I_0 - rI_1)\right] , \qquad (107)$$

where the modified Bessel functions I_0 and I_1 have argument $\left(\frac{r\rho\sin\theta}{2L}\right)$. For the current given in Eq. (100) and conductivity given in Eq. (101), we obtain the electric field corresponding to $\mathbf{B}$ in Eq. (105):

$$\mathbf{E} = -\frac{J_0 R_0{}^3}{\sigma_0 r^2}\,\delta(r - ct)e^{-\frac{r}{\lambda}}\,\hat{\mathbf{r}}\,e^{\frac{t}{t_0}}$$

$$+ \frac{\mu J_0 R_0{}^3}{t_0^2 4\sqrt{\pi}}\left(\frac{t}{d}\right)\int_{0}^{1}\frac{dx}{\ell^{5/2}} \exp\left(\frac{t}{t_0}(x - 1) - ct\,\frac{x}{\lambda}\right)\exp -\left(\frac{r^2 + c^2t^2x^2}{4d^2\ell}\right)$$

$$\cdot \left[\hat{\mathbf{r}}\cos\theta I_0 - \hat{\boldsymbol{\theta}}\left(\sin\theta I_0 - \frac{r}{ctx}I_1\right)\right] . \qquad (108)$$

where I_0 and I_1 have argument $\left(\frac{rctx\sin\theta}{2d^2\ell}\right)$. We can obtain the $\mathbf{E}$-field solution for constant conductivity corresponding to $\mathbf{B}_c$ by setting $t_0 \to \infty$ so that $\sigma(t) = \sigma_0$ in Eq. (108). We find

$$\mathbf{E}_c = -\frac{J_0 R_0^3}{\sigma_0 r^2}\,\delta(r - ct)e^{-\frac{r}{\lambda}}\,\hat{r}$$

$$+ \frac{\mu J_0 R_0^3}{4\sqrt{\pi}\,t}\left(\frac{\mu\sigma_0}{t}\right)^{1/2}\int_0^1 \frac{dx}{(1-x)^{5/2}}\exp\left(-\frac{ctx}{\lambda}\right)\exp\left[-\frac{\mu\sigma_0}{4t}\frac{\left(r^2 + c^2t^2x^2\right)}{(1-x)}\right]$$

$$\cdot\left[\hat{r}\cos\theta I_0 - \hat{\theta}\left(\sin\theta I_0 - \frac{r}{ctx}I_1\right)\right], \qquad (109)$$

where I_0 and I_1 have arguments $\left(\frac{\mu\sigma_0 rcx\sin\theta}{2(1-x)}\right)$.

In a subsequent paper, we will present numerical results obtained from all of these solutions along with a comparison of those numerical results with solutions obtained by other methods.

APPENDIX A
CHECK OF INFINITE CONDUCTIVITY SOLUTION

We now check explicitly that our procedure has actually manufactured a solution to the Maxwell's equations and boundary conditions appropriate to the infinite ground conductivity case. First, the fact that Eq. (14) gives **B** as a perfect curl implies that **B** is divergenceless everywhere above the ground surface:

$$\nabla \cdot \mathbf{B}_{>} = - \nabla \cdot \nabla \times \int_{>} \frac{d^4x'}{\sigma'} (G_0 \mathbf{J} - G_I \tilde{\mathbf{J}}) = 0 . \qquad (110)$$

Then, the properties of the Green's functions in Eq. (14) show immediately that **B** solves the diffusion equation (with time-dependent conductivity and appropriate source) everywhere above the ground:

$$\left(\frac{\nabla^2}{\mu\sigma_{>}} - \partial_t\right) B_{>} = - \nabla \times \int_{>} \frac{d^4x'}{\sigma'} \left[\delta(\mathbf{x} - \mathbf{x}')\delta(t - t')J - \delta(\mathbf{x} - \tilde{\mathbf{x}}')\delta(t - t')\tilde{\mathbf{J}}\right]$$

$$= - \nabla \times \left[\theta(z) \frac{\mathbf{J}}{\sigma_{>}} - \theta(-z) \frac{\mathbf{J}}{\sigma_{<}}\right] = - \nabla \times \frac{\mathbf{J}}{\sigma_{>}} \quad \text{for} \quad z > 0 . \qquad (111)$$

Since **E** given by Eqs. (18) or (19) was obtained from **B** through Ampere's law, **E** and **B** given by Eqs. (19) and (14) satisfy

$$\mathbf{E}_{>} + \frac{\mathbf{J}}{\sigma_{>}} = \frac{\nabla \times \mathbf{B}}{\mu\sigma_{>}} , \qquad (112)$$

by construction. As a consequence, **E** and **B** are also related by

$$\nabla \times \mathbf{E}_{>} = \nabla \times \frac{1}{\sigma_{>}} \int_{>} d^4x' [G_0 \dot{\mathbf{J}} - G_I \dot{\tilde{\mathbf{J}}}] = \nabla \times \int_{>} \frac{d^4x'}{\sigma'} [\partial_t G_0 \mathbf{J} - \partial_t G_I \tilde{\mathbf{J}}] = - \dot{\mathbf{B}}_{>} . \qquad (113)$$

Finally, we may calculate the divergence of Eq. (19) to identify the electric charge density that accompanies our fields. We find:

$$\nabla \cdot \mathbf{E}_{>} = \frac{1}{\sigma_{>}} \int_{>} d^4x' \; \nabla \cdot [G_0 \dot{\mathbf{J}} - G_I \dot{\tilde{\mathbf{J}}}] - \frac{\nabla^2}{\mu\sigma_{>}} \int_{>} \frac{d^4x'}{\sigma'} G_D \nabla' \cdot \mathbf{J}$$

$$= \frac{1}{\sigma_{>}} \int_{>} d^4x' \; G_D \nabla' \cdot \dot{\mathbf{J}} - \frac{\nabla^2}{\mu\sigma_{>}} \int_{>} \frac{d^4x'}{\sigma'} G_D \nabla' \cdot \mathbf{J}$$

$$= \left(\partial_t - \frac{\nabla^2}{\mu\sigma_{>}} \right) \int_{>} \frac{d^4x'}{\sigma'} G_D \nabla' \cdot \mathbf{J} = - \nabla \cdot \frac{\mathbf{J}}{\sigma_{>}} , \tag{114}$$

everywhere above the ground. [This agrees with the divergencelessness of the total current (Ohmic plus Compton) implied by Ampere's law in our approximation.]

We must now check that the boundary behavior of **E** and **B** given by Eqs. (19) and (14) is consistent with our problem. First, Eq. (19) implies that tangential **E** is given by:

$$\mathbf{E}_T = \frac{1}{\sigma_{>}} \int_{>} d^4x' \; G_D \dot{\mathbf{J}}_T - \frac{\nabla_T}{\mu\sigma_{>}} \int_{>} \frac{d^4x'}{\sigma'} G_D \nabla' \cdot \mathbf{J} . \tag{115}$$

Then the fact that both the Dirichlet Green's function and its tangential gradient vanish as $z \to 0$ implies that

$$\mathbf{E}_T \Big|_{z=0} = 0 . \tag{116}$$

E_z, on the other hand, is given by a superposition of the Neumann Green's function and the normal derivative of the Dirichlet Green's function:

$$E_z = \frac{1}{\sigma_>} \int_> d^4x' \; G_N \dot{J}_z - \frac{\partial_z}{\mu\sigma_>} \int_> \frac{d^4x'}{\sigma'} G_D \; \nabla' \cdot J \; , \qquad (117)$$

neither of which vanishes on the surface. In fact,

$$E_z\Big|_{z=0} = 2\left(\frac{1}{\sigma_>} \int_> d^4x' \; G_0 \dot{J}_z - \frac{1}{\mu\sigma_>} \int_> \frac{d^4x'}{\sigma'} \partial_z G_0 \; \nabla' \cdot J \right)_{z=0} \; . \qquad (118)$$

so that at the ground surface **E** is entirely vertical with double its free-space value. Since **E** vanishes everywhere in the ground, this implies a surface charge density

$$\sum = 2\epsilon\left(\frac{1}{\sigma_>} \int_> d^4x' \; G_0 \dot{J}_z - \frac{1}{\mu\sigma_>} \int_> \frac{d^4x'}{\sigma'} \partial_z G_0 \; \nabla' \cdot J \right)_{z=0} \qquad (119)$$

at the ground surface.

Equation (14) implies that B_z depends only on the tangential derivatives of the Dirichlet Green's function:

$$B_z = \mathbf{k} \cdot \nabla \times \int_> \frac{d^4x'}{\sigma'} \left(G_0 \mathbf{J} - G_I \tilde{\mathbf{J}}\right) = \mathbf{k} \cdot \nabla_T \times \int_> \frac{d^4x'}{\sigma'} G_D \mathbf{J}_T \; . \qquad (120)$$

Consequently,

$$B_z\Big|_{z=0} = 0 \; . \qquad (121)$$

Since $\dot{\mathbf{B}}$ is just $-\nabla \times \mathbf{E}$ so that **B** vanishes everywhere in the ground, Eq. (120) implies that our B_z is continuous at the ground surface so that **B** is divergenceless everywhere. Finally, the transverse part of **B** [according to Eq. (14)] is just:

$$\mathbf{B}_T = -\nabla_T \times \int_> \frac{d^4x'}{\sigma'} G_N J_z \mathbf{k} - \mathbf{k}\partial_z \times \int_> \frac{d^4x'}{\sigma'} G_D \mathbf{J}_T , \tag{122}$$

so that at the surface

$$\mathbf{B}_T\Big|_{z=0} = -2\left(\nabla_T \times \int_> \frac{d^4x'}{\sigma'} G_0 J_z \mathbf{k} + \mathbf{k}\partial_z \times \int_> \frac{d^4x'}{\sigma'} G_0 \mathbf{J}_T\right)_{z=0} , \tag{123}$$

or

$$\mathbf{B}_T\Big|_{z=0} = -2\left(\nabla \times \int_> \frac{d^4x'}{\sigma'} G_0 \mathbf{J}\right)_{z=0,T} . \tag{124}$$

Thus, **B** is entirely transverse with double its free-space value at the ground surface. Since **B** vanishes everywhere below the ground surface, this implies a surface current at the ground surface given by:

$$\mathbf{K} = -\frac{2}{\mu} \mathbf{k} \times \left(\nabla \times \int_> \frac{d^4x'}{\sigma'} G_0 \mathbf{J}\right)_{z=0} . \tag{125}$$

APPENDIX B

SOLUTION OF INTEGRAL EQUATION USING INTEGRAL IDENTITIES

In Appendix A of Van Alstine and Schlessinger [1986],* we applied Green's theorem to two Green's functions to obtain the integral identity:

$$G_I(\mathbf{x}, \mathbf{x}'') = -2 \int \frac{dt'}{\mu\sigma'} \int dS' \left[\partial_{z'} G_0(\mathbf{x}, \mathbf{x}') G_0(\mathbf{x}', \mathbf{x}'') \right]_{z'=0^+} . \qquad (126)$$

The subscript $z'=0^+$ indicates that the surface integral is to be viewed as the boundary of a volume integral in the upper half space $z' \geq 0$. By differentiating this identity and using the facts that $G_0 = G_0(\mathbf{x} - \mathbf{x}')$ while $G_I = G_I(\mathbf{x} - \tilde{\mathbf{x}}')$, we find:

$$\tilde{\nabla} G_I(\mathbf{x}, \mathbf{x}'') = - \nabla'' G_I(\mathbf{x}, \mathbf{x}'')$$

$$= 2 \int \frac{dt'}{\mu\sigma'} \int dS' \left[\partial_{z'} G_0(\mathbf{x}, \mathbf{x}') \nabla'' G_0(\mathbf{x}', \mathbf{x}'') \right]_{z'=0^+} , \qquad (127)$$

which becomes

$$\tilde{\nabla} G_I(\mathbf{x}, \mathbf{x}'') = -2 \int \frac{dt'}{\mu\sigma'} \int dS' \left[\partial_{z'} G_0(\mathbf{x}, \mathbf{x}') \nabla' G_0(\mathbf{x}', \mathbf{x}'') \right]_{z'=0^+} . \qquad (128)$$

Because on the surface $z''=0$ $G_I = G_0$ and $\partial_z G_I = \partial_z G_0$, Eqs. (126) and (128) become

---

*Van Alstine, P., and L. Schlessinger, Source Region Electromagnetic Effects Phenomena, Vol. 4, New Methods for Determination of Three-Dimensional SREMP Environments, Pacific-Sierra Research Corporation, Report 1588, December 1986.

$$G_0(x, x'')\Big|_{z''=0} = -2\int\frac{dt'}{\mu\sigma'}\int dS'\left[\partial_{z'}G_0(x, x')G_0(x', x'')\right]_{z'=0^+,z''=0} \tag{129}$$

and

$$\widetilde{\nabla}G_0(x, x'')\Big|_{z''=0} = -2\int\frac{dt'}{\mu\sigma'}\int dS'\left[\partial_{z'}G_0(x, x')\nabla' G_0(x', x'')\right]_{z'=0^+,z''=0} \tag{130}$$

respectively.

Now, Eqs. (129) and (130) show that integration over time and surface of $\partial_{z'}G_0|_{z'=0^+}$, with either G_0 or its gradient evaluated with both of its spatial arguments on the surface, essentially reproduces the Green's function or its gradient with its first spatial argument moved off the surface. Such an operation would take the Ω term in the integral equation Eq. (25) (whose Green's functions are fully evaluated on the surface) and turn it directly into the Ω term in Eq. (7) (whose first spatial argument roams over the upper half-space). If we write Eq. (25) out in full, we see that

$$\begin{aligned} 2\lim_{z\to 0^+}\int_{-\infty}^{t} dt'\Bigg[&\int dS' k \times E(x')G_0(x, x')\Big|_{z'=0} \\ &- \nabla_T\nabla_T\cdot\int dS' k \times E(x')H(x, x')\Big|_{z'=0}\Bigg] \\ = -\Bigg[&\lim_{z\to 0^+}\nabla \times \int_{>}\frac{d^4x'}{\sigma'}G_0 J - \lim_{z\to 0^-}\nabla \times \int_{<}\frac{d^4x'}{\sigma'}G_0 J\Bigg] \end{aligned} \tag{131}$$

so that we need to deal with two further complications if we are to use our integral identities. The first is that our integral identities involve two Green's functions both of which are boundary values from above while the second term on the right-hand side of Eq. (131) contains a Green's function evaluated on the surface from below. Therefore, we rewrite it:

$$\lim_{z\to 0^-} \nabla \times \int_< \frac{d^4x'}{\sigma'} G_0 J = \lim_{z\to 0^+} \tilde{\nabla} \times \int_< \frac{d^4x'}{\sigma'} G_I J_<(x')$$

$$= \lim_{z\to 0^+} \tilde{\nabla} \times \int_> \frac{d^4x'}{\sigma'} G_0 J_<(\tilde{x}') \ . \qquad (132)$$

Then, Eq. (131) becomes:

$$2 \lim_{z\to 0^+} \int_{-\infty}^{t} dt' \Bigg[\int dS' k \times E(x') G_0(x, x')\Big|_{z'=0}$$

$$- \nabla_T \nabla_T \cdot \int dS' k \times E(x') H(x, x')\Big|_{z'=0} \Bigg]$$

$$= - \left[\lim_{z\to 0^+} \nabla \times \int_> \frac{d^4x'}{\sigma'} G_0 J_> - \lim_{z\to 0^+} \tilde{\nabla} \times \int_> \frac{d^4x'}{\sigma'} G_0 J_<(\tilde{x}') \right] . \qquad (133)$$

Secondly, we have to check that H in the second term on the left-hand side of Eq. (133) obeys its own version of Eq. (129):

$$-2 \int \frac{dt'}{\mu\sigma'} \int dS' \left[\partial_{z'} G_0(x, x') H(x', x'', t', t'') \right]_{z'=0^+, z''=0}$$

$$= - 2 \int \frac{dt'}{\mu\sigma'} \int dS' \left[\partial_{z'} G_0(x, x') \int_{t''}^{\infty} \frac{dt'''}{\mu\sigma'''} G_0(x', x'', t', t''') \right]_{z'=0^+, z''=0}$$

$$= \int_{t''}^{\infty} \frac{dt'''}{\mu\sigma'''} G_0(x, x'', t, t''')\Big|_{z''=0} = H(x, x'', t, t'')\Big|_{z''=0} \ . \qquad (134)$$

In short, if one inspects the identities [Eqs. (126), (128), (129), (130), (134)], one sees that the net result of multiplying each term in Eq. (133) by $-2\,\partial_z G_0(z'', z)|_{z=0^+}$ and integrating over $\int dt/\mu\sigma \int dS$ will be to shift the first z argument of all terms from $z=0^+$ to variable z and to change ∇G_0 into $\tilde{\nabla} G_I$ and $\tilde{\nabla} G_0$ into ∇G_I on the right-hand side. That is, performing this operation on Eq. (133) yields:

$$2 \int_{-\infty}^{t} dt' \boldsymbol{\Omega} \cdot (\mathbf{k} \times \mathbf{E})_T = -\left[\tilde{\nabla} \times \int_{>} \frac{d^4x'}{\sigma'} G_I \mathbf{J}_> - \nabla \times \int_{>} \frac{d^4x'}{\sigma'} G_I \mathbf{J}_<(\tilde{\mathbf{x}}')\right]_T , \tag{135}$$

in which z is variable. Setting $z' \rightarrow -z'$ in the second term on the right-hand side of Eq. (135), we see that:

$$2 \int_{-\infty}^{t} dt' \boldsymbol{\Omega} \cdot (\mathbf{k} \times \mathbf{E})_T = -\left[\tilde{\nabla} \times \int_{>} \frac{d^4x'}{\sigma'} G_I \mathbf{J}_> - \nabla \times \int_{<} \frac{d^4x'}{\sigma'} G_0 \mathbf{J}_<(\mathbf{x}')\right]_T . \tag{136}$$

Then, the transverse part of the identity Eq. (6) tells us that

$$(\tilde{\mathbf{A}} \times \mathbf{B})_T = -(\mathbf{A} \times \tilde{\mathbf{B}})_T \tag{137}$$

so that the first term on the right-hand side becomes:

$$\left(\tilde{\nabla} \times \int_{>} \frac{d^4x'}{\sigma'} G_I \mathbf{J}_>\right)_T = -\left(\nabla \times \int_{>} \frac{d^4x'}{\sigma'} G_I \tilde{\mathbf{J}}_>\right)_T . \tag{138}$$

Thus, Eq. (135) may be rewritten as

$$2 \int_{-\infty}^{t} dt' \boldsymbol{\Omega} \cdot (\mathbf{k} \times \mathbf{E})_T = +\left(\nabla \times \int_{>} \frac{d^4x'}{\sigma'} G_I \tilde{\mathbf{J}}_> + \nabla \times \int_{<} \frac{d^4x'}{\sigma'} G_0 \mathbf{J}_<\right)_T . \tag{139}$$

DISTRIBUTION LIST

DEPARTMENT OF DEFENSE

DEFENSE INTELLIGENCE AGENCY
ATTN: DT-1B
2 CYS ATTN: RTS-2B
ATTN: VP-TPO

DEFENSE NUCLEAR AGENCY
ATTN: DFRA
ATTN: RAAE
ATTN: RAEE
ATTN: RAEV
4 CYS ATTN: TITL

DEFENSE TECH INFO CENTER
12CYS ATTN: DD

LAWRENCE LIVERMORE NATIONAL LAB
ATTN: DNA-LL

DEPARTMENT OF THE ARMY

HARRY DIAMOND LABORATORIES
ATTN: CHIEF DIV 10000
ATTN: SCHLD-NW-P, J BRAND
ATTN: SLCHD-NW
2 CYS ATTN: SLCHD-NW-E
ATTN: SLCHD-NW-EA
ATTN: SLCHD-NW-EE
ATTN: SLCHD-NW-RA
ATTN: SLCHD-NW-RC
ATTN: SLCHD-NW-RH G MERKEL
ATTN: SLCHD-NW-RH R GILBERT
ATTN: SLCHD-NW-RI G HUTTLIN
ATTN: SLCIS-IM-TL (TECH LIB)

RESEARCH & DEV CENTER
ATTN: DRCPM-TDS-SD

U S ARMY ATMOSPHERIC SCIENCES LAB
ATTN: SLCAS-AS

U S ARMY BALLISTIC RESEARCH LAB
ATTN: SLCBR-SS-T (TECH LIB)
ATTN: SLCBR-VL

U S ARMY BELVOIR RD & E CTR
ATTN: STRBE-BT (TECH LIB) (VAULT)

U S ARMY ENGINEER DIV HUNTSVILLE
ATTN: HNDED-SY

U S ARMY MISSILE COMMAND
ATTN: AMSMI-RD-GC-P

U S ARMY MSL COMMAND/AMSMI-RD-CS-R
ATTN: AMSMI-RD-CS-R (DOCS)

U S ARMY NUCLEAR & CHEMICAL AGENCY
ATTN: LIBRARY

DEPARTMENT OF THE NAVY

NAVAL POSTGRADUATE SCHOOL
ATTN: CODE 1424

NAVAL RESEARCH LABORATORY
ATTN: CODE 2627

NAVAL SURFACE WARFARE CENTER
ATTN: CODE R40
ATTN: CODE R43
ATTN: CODE 425

DEPARTMENT OF THE AIR FORCE

AERONAUTICAL SYSTEMS DIVISION
ATTN: ASD/ENACE
ATTN: ASD/ENES(P MARTH)

AIR FORCE INSTITUTE OF TECH/EN
ATTN: LIBRARY/AFIT/LDEE

AIR FORCE WEAPONS LABORATORY
ATTN: NTAA
ATTN: NTYC
ATTN: SUL

AIR UNIVERSITY LIBRARY
ATTN: AUL-LSE

ROME AIR DEVELOPMENT CENTER, AFSC
ATTN: TECHNICAL LIBRARY (DOVL)

SPACE DIVISION/YA
ATTN: YAR

DEPARTMENT OF ENERGY

LAWRENCE LIVERMORE NATIONAL LAB
ATTN: D OAKLEY
ATTN: TECH INFO DEPT. LIBRARY

LOS ALAMOS NATIONAL LABORATORY
ATTN: REPORT LIBRARY

SANDIA NATIONAL LABORATORIES
ATTN: TECH LIB 3141

OTHER GOVERNMENT

CENTRAL INTELLIGENCE AGENCY
ATTN: OSWR/NED
ATTN: OSWR/STD/MTB

DEPARTMENT OF DEFENSE CONTRACTORS

AGBABIAN ASSOCIATES
ATTN: LIBRARY

ALLIED-SIGNAL, INC
ATTN: DOCUMENT CONTROL

AT&T TECHNOLOGIES, INC
ATTN: W EDWARDS

BDM CORP
ATTN: CORPORATE LIB

BDM CORP
ATTN: A RIGGS
ATTN: LIBRARY

BOEING CO
ATTN: D EGELKROUT
ATTN: H WICKLEIN
ATTN: R SCHEPPE

BOEING MILITARY AIRPLANE CO
ATTN: C SUTTER

BOOZ-ALLEN & HAMILTON, INC
ATTN: TECHNICAL LIBRARY

BOOZ-ALLEN & HAMILTON, INC
ATTN: L ALBRIGHT

BOOZ-ALLEN & HAMILTON, INC
ATTN: D DURGIN

CALSPAN CORP
ATTN: LIBRARY

COMPUTER SCIENCES CORP
ATTN: A SCHIFF

E-SYSTEMS, INC
ATTN: MAIL STOP 3

E-SYSTEMS, INC
ATTN: J MOORE

EG&G WASH ANALYTICAL SVCS CTR, INC
ATTN: A BONHAM
ATTN: J GILES

ELECTRO-MAGNETIC APPLICATIONS, INC
ATTN: D MEREWETHER

GENERAL ELECTRIC CO
ATTN: C HEWISON

GENERAL ELECTRIC COMPANY
ATTN: J PALCHEFSKY, JR

GENERAL RESEARCH CORP
ATTN: J WYSS

GRUMMAN AEROSPACE CORP
ATTN: SMITH, TECH INFO CNTR

GTE GOVERNMENT SYSTEMS CORPORATION
ATTN: TECH LIBRARY

HERCULES DEF ELECT SYSTEMS, INC
ATTN: R LAZARCHIK

HERCULES, INC
ATTN: W WOODRUFF

HONEYWELL, INC
ATTN: LIBRARY

HONEYWELL, INC
ATTN: SR&C LIBRARY/T CLARKIN

IIT RESEARCH INSTITUTE
ATTN: I MINDEL

INSTITUTE FOR DEFENSE ANALYSES
ATTN: CLASSIFIED LIBRARY
ATTN: TECH INFO SERVICES

IRT CORP
ATTN: B WILLIAMS
ATTN: R W STEWART

ITT TELECOMMUNICATIONS CORP
ATTN: R SCHWALLIE

JAYCOR
ATTN: LIBRARY

JOHNS HOPKINS UNIVERSITY
ATTN: P PARTRIDGE

KAMAN SCIENCES CORP
ATTN: K LEE

KAMAN SCIENCES CORP
ATTN: LIBRARY/B KINSLOW

KAMAN SCIENCES CORP
ATTN: E CONRAD

KAMAN SCIENCES CORPORATION
ATTN: TECHNICAL LIBRARY

KAMAN SCIENCES CORPORATION
ATTN: DASIAC

KAMAN TEMPO
ATTN: DASIAC
ATTN: R RUTHERFORD

LITTON SYSTEMS, INC
ATTN: E EUSTIS

LITTON SYSTEMS, INC
ATTN: J SKAGGS

LOCKHEED MISSILES & SPACE CO, INC
ATTN: J PEREZ

LOCKHEED MISSILES & SPACE CO, INC
ATTN: TECH INFO CTR D/COLL

LTV AEROSPACE & DEFENSE COMPANY
ATTN: LIBRARY

LUTECH, INC
ATTN: TOM LIV

MCDONNELL DOUGLAS CORP
ATTN: TECHNICAL LIBRARY

METATECH CORPORATION
ATTN: W RADASKY

MISSION RESEARCH CORP
ATTN: EMP GROUP

MISSION RESEARCH CORP
ATTN: J LUBELL
ATTN: J R CURRY

MISSION RESEARCH CORP, SAN DIEGO
ATTN: V VAN LINT

MITRE CORPORATION
ATTN: M FITZGERALD

PACIFIC-SIERRA RESEARCH CORP
ATTN: H BRODE, CHAIRMAN SAGE
2 CYS ATTN: L SCHLESSINGER
2 CYS ATTN: P VAN ALSTINE

PHOTOMETRICS, INC
ATTN: I L KOFSKY

PHYSICS INTERNATIONAL CO
ATTN: DOCUMENT CONTROL

R & D ASSOCIATES
ATTN: DOCUMENT CONTROL

R & D ASSOCIATES
ATTN: LIBRARY

R & D ASSOCIATES
ATTN: J P CASTILLO

RAYTHEON CO
ATTN: H FLESCHER

RCA CORPORATION
ATTN: G BRUCKER

RESEARCH TRIANGLE INSTITUTE
ATTN: M SIMONS

ROCKWELL INTERNATIONAL CORP
ATTN: B-1 DIV TIC (BAOB)

ROCKWELL INTERNATIONAL CORP
ATTN: G MORGAN

S-CUBED
ATTN: A WILSON

SCIENCE & ENGRG ASSOCIATES
ATTN: B GAGE
ATTN: B ZETLEN
ATTN: M BORDEN

SCIENCE & ENGRG ASSOCIATES, INC
ATTN: V JONES

SCIENCE APPLICATIONS INTL CORP
ATTN: W CHADSEY

SCIENCE APPLICATIONS INTL CORP
ATTN: E ODONNELL
ATTN: P J DOWLING

SINGER CO
ATTN: TECH INFO CENTER

SRI INTERNATIONAL
ATTN: D ARNS

TELEDYNE BROWN ENGINEERING
ATTN: F LEOPARD

TEXAS INSTRUMENTS, INC
ATTN: TECHNICAL LIBRARY

TRW INC
ATTN: LIBRARIAN

TRW INC
ATTN: A R CARLSON
ATTN: J PENAR

UNISYS CORP-DEFENSE SYSTEMS
ATTN: TECHNICAL LIBRARY